Can quantum correlations be completely quantum?

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Abstract
Deficit of information zero-way was proposed in Horodecki (2005 *Phys. Rev. A* **71** 062307) as one of the possible measures of quantumness of correlations. Numerical calculations suggested that there exist such states for which this quantity is almost equal to mutual information. In this paper, we present a family of states for which we have equality between the above measure of quantumness of correlations and the measure of total correlations—mutual information. This means that whole correlations in these states have, in some sense, quantum character and that quantum correlations do not necessarily imply classical correlations. We prove this intriguing feature for a subclass of $2 \otimes 2$ separable states. We also present numerical results suggesting that this interesting situation might also happen for $2 \otimes 2$ entangled states.

PACS numbers: 03.65.Ud, 03.67.−a

(Some figures in this article are in colour only in the electronic version)

1. Introduction

It has been found in different contexts [1–5] that entanglement does not exhaust quantumness of correlations contained in compound quantum systems. In order to characterize quantumness of states, the *properly classically correlated* bipartite states (shortly classically correlated) were defined [5]. These states can be written in the form

$$\rho_{AB} = \sum_{ij} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|, \tag{1}$$

with coefficients $0 \leqslant p_{ij} \leqslant 1$, $\sum_{ij} p_{ij} = 1$; $\{|i\rangle\}$ and $\{|j\rangle\}$ are local bases. The set of classically correlated states is invariant under local unitary operations. These states are diagonal in the so-called byproduct basis. The *non-classically* correlated states are those that cannot be represented in the above form.

In [5] a measure of quantumness of correlations the so-called quantum deficit Δ was introduced, which is zero for classically correlated states (see extensive development [1]). In particular, it has been shown that there exist separable states which have nonzero deficit. This

means that those separable states exhibit classical correlations between quantum properties. Paper [1] introduces variants of quantum deficit with restricted communication as independent candidates for the measure of quantumness of correlations. One of them called the quantum deficit zero-way $\Delta^\emptyset$ is equivalent to the distance from classically correlated states [1]. Recently, similar measures of quantumness of correlations based on the distance from classically correlated states have been introduced [6, 7] and quantumness of some subclasses of separable and entangled states was investigated.

The main purpose of the present paper is to inquire a relation between the measure of total correlations—mutual information and quantum deficit. The latter is defined [5] as the difference between the informational content of a state and information that can be localized to a subsystem by the use of local unitary operations and a dephasing channel. Quantum deficit refers us to this part of correlations that must be destroyed during the process of localizing information into a subsystem. In this paper, we consider a special kind of quantum deficit called zero-way ($\Delta^\emptyset$) [1] where only a restricted class, called $\mathcal{P}^\emptyset$, of protocols of localizing information is permitted. In $\mathcal{P}^\emptyset$ protocols we make local measurements and only after this we use a classical channel to collect data and then exploit classical correlations created by the measurements to localize information. Surprisingly, we find such states for which the quantum deficit zero-way is equal to the mutual information I_M (the measure of total correlations in a state), i.e. $\Delta^\emptyset = I_M$. This means that all correlations in these states have quantum character. Note that for maximally entangled states $\Delta^\emptyset = \frac{1}{2} I_M$, so we can say that only half of the correlations manifest quantumness. In this context, the $\Delta^\emptyset = I_M$ feature of the states presented in this paper seems especially interesting.

One could ask whether similar equality could also hold in the case of quantum deficit Δ, i.e. if there exist states for which $\Delta = I_M$? It is conjectured in [1] that such states do not exist and it is even not known whether there are states for which $\Delta > \frac{1}{2} I_M$.

The notion of quantum deficit is built by the use of quantities which has operational meaning in the regime of many copies: information I is the number of pure states we can distill from a given state by the so-called noisy operations; localizable information I_l is equal to the number of pure product states we can distill by CLOCC (closed LOCC [1]) operation in the asymptotic regime. These two quantities can also be interpreted in terms of the work we can draw from a heat bath by the use of a given quantum state. Quantum deficit in the asymptotic regime inherits this clean operational meaning. But, on the other hand, it is much harder to evaluate. Some bounds on the regularized quantum deficit have been obtained in [8]. In our case, we consider one copy version of quantum deficit and additionally allow only for a special kind of protocol of localizing information into a subsystem. In spite of such strong restrictions we get a quite interesting result.

2. Basic notion

Informational content I of a state. Information is an abstract concept. Here we will use this term to refer to a specific function I

$$I(\varrho) = N - S(\varrho),\qquad(2)$$

where ϱ is a state of N qubits and $S(\varrho) = -\mathrm{Tr}\,\varrho \log \varrho$ is the von Neumann entropy. The information I has operational meaning in the asymptotic regime of many identical copies [9]. This is the unique function (up to constants) that is not increasing under the class of so-called noisy operations (NO) [10]: operations that consist of (i) unitary transformations, (ii) partial trace and (iii) adding ancilla in maximally mixed state. One can then show that I determines the optimal rate of transitions between states under NO.

By CLOCC operations on bipartite system of n_{AB} qubits we mean all operations that can be composed out of

  (i) local unitary transformations and
 (ii) sending a subsystem down completely the decohering (dephasing) channel.

Note that the CLOCC class is a subclass of LOCC operations (local operations and classical communication) and is equivalent to local measurements, local unitary operations and classical communication. The restrictions of CLOCC (in contrast to LOCC we are forbidden to add pure ancillas and discard a local subsystem) are a consequence of the fact that we must control information flow. Every state (besides maximally mixed ones) has nonzero value of I, so especially pure ancillas.

Localizable information $I_l(\varrho_{AB})$ of a state ϱ_{AB} is the maximal amount of local information that can be obtained by CLOCC operations,

$$I_l(\varrho_{AB}) = \sup_{\Lambda \in CLOCC} (I(\varrho'_A) + I(\varrho'_B)), \tag{3}$$

where $\varrho'_{AB} = \Lambda(\varrho_{AB})$.

Local information I_{LO} is the difference between the number of qubits N of a state and the sum of entropies of its subsystems:

$$I_{LO}(\varrho_{AB}) = N - S_A(\varrho_{AB}) - S_B(\varrho_{AB}). \tag{4}$$

The *mutual information* I_M of a state ϱ_{AB} is the difference between the sum of local entropies and the entropy of the state:

$$I_M(\varrho_{AB}) = S_A(\varrho_{AB}) + S_B(\varrho_{AB}) - S(\varrho_{AB}). \tag{5}$$

3. Quantum and classical deficit zero-way

In this section, we introduce the quantities which are fundamental for our consideration—quantum deficit and classical deficit.

Definition 1. *The quantum deficit* $\Delta(\varrho_{AB})$ *of a state* ϱ_{AB} *is given by the difference between the informational content I of the state and the localizable information I_l:*

$$\Delta(\varrho_{AB}) = I(\varrho_{AB}) - I_l(\varrho_{AB}). \tag{6}$$

Note that the quantum deficit can be rewritten as

$$\Delta(\varrho_{AB}) = \inf_{\Lambda \in CLOCC} S(\varrho'_{AB}) - S(\varrho_{AB}), \tag{7}$$

where $\varrho'_{AB} = \Lambda(\varrho_{AB})$ and Λ is optimized over CLOCC operations.

Quantum deficit tells us about the amount of information that cannot be localized into a subsystem. This means that a part of information is necessarily destroyed in the process of localizing information by the use of a classical channel. So this part must be somehow quantum and come from correlations. This is the reason why we interpret quantum deficit as a measure of quantumness of correlations.

Definition 2. *The classical deficit* $\Delta_{cl}(\varrho_{AB})$ *of a quantum state is the difference between the information that can be localized by means of CLOCC operations (i.e. localizable information I_l) and local information*

$$\Delta_{cl}(\varrho_{AB}) = I_l(\varrho_{AB}) - I_{LO}(\varrho_{AB}). \tag{8}$$

Classical deficit tells us how much more information can be obtained from a state ϱ_{AB} by exploiting additional correlations in the state using a classical channel.

Note that the quantum and classical deficit add up to the mutual information I_M:

$$\Delta + \Delta_{cl} = I_M. \tag{9}$$

Thus we can express classical deficit as follows:

$$\Delta_{cl} = I_M - \Delta. \tag{10}$$

We can restrict the classical communication between Alice and Bob to one-way communication (from Alice to Bob or from Bob to Alice) or to the so-called zero-way communication.

Consider the zero-way protocol $\mathcal{P}^\emptyset$ of localizing information consisting of complete local measurement (or local complete dephasing) and classical communication which is allowed to be performed only after making measurement (or dephasing). Classical communication is necessary to collect data and exploit the pure classical correlations in order to localize information. Note that the main difference between a zero-way protocol and a general CLOCC protocol of localizing information is that we are forbidden to communicate classically before all measurement or dephasing operations are finished. This guarantees that we are not able to draw any nonclassical information from correlations in a state, for example, from correlations of subsystems of some separable states which we can call classical correlations between quantum properties of the state.

If we restrict classical communication in the localizable information I_l to zero-way protocols $\mathcal{P}^\emptyset$, we get the localizable information zero-way $I_l^\emptyset$

$$I_l^\emptyset(\varrho_{AB}) = \sup_{\Lambda \in \mathcal{P}^\emptyset} (I(\varrho'_A) + I(\varrho'_B)). \tag{11}$$

Now we can define the quantum deficit zero-way $\Delta^\emptyset$ and the classical deficit zero-way $\Delta_{cl}^\emptyset$.

Definition 3. *The quantum deficit zero-way $\Delta^\emptyset(\varrho_{AB})$ of a state ϱ_{AB} is given by the difference between the informational content I of the state and the information $I_l^\emptyset$ localizable by the zero-way protocol $\mathcal{P}^\emptyset$:*

$$\Delta^\emptyset(\varrho_{AB}) = I(\varrho_{AB}) - I_l^\emptyset(\varrho_{AB}). \tag{12}$$

Equivalently, we can express the quantum deficit zero-way as

$$\Delta^\emptyset(\varrho_{AB}) = \inf_{\Lambda \in \mathcal{P}^\emptyset} S(\varrho'_{AB}) - S(\varrho_{AB}), \tag{13}$$

where $\varrho'_{AB} = \Lambda(\varrho_{AB})$ and Λ is optimized over $\mathcal{P}^\emptyset$ protocols. One can also find that the quantum deficit zero-way $\Delta^\emptyset$ is equal to the relative entropy distance from the set of classically correlated states [1].

The quantum deficit zero-way is an independent candidate for the measure of quantumness of correlations. States having no quantum correlations should have $\Delta^\emptyset = 0$. Such a restricted measure can capture interesting aspects of nonlocality. Consider, for example, a bipartite state with the eigenbasis of the form

$$|0\rangle_A|0\rangle_B, \quad |0\rangle_A|1\rangle_B, \qquad \tfrac{1}{\sqrt{2}}|1\rangle_A(|0\rangle + |1\rangle)_B, \quad \tfrac{1}{\sqrt{2}}|1\rangle_A(|0\rangle - |1\rangle)_B; \tag{14}$$

such a basis is locally indistinguishable and is also not distinguishable by zero-way communication. Therefore, a mixture of the states (14) where the mixing probabilities are all

different from each other would have nonvanishing $\Delta^\emptyset$ [1]. This is in contrast to states which are mixtures of the set of states

$$|0\rangle_A |0\rangle_B, |0\rangle_A |1\rangle_B, |1\rangle_A |0\rangle_B, |1\rangle_A |1\rangle_B$$

for which all the information is extractable from the state locally, by measurement by both the parties without any communication. It suggests that the quantum behavior of correlations could result from distinctly quantum but 'local' properties of nonorthogonality. This is connected with examples of LOCC-indistinguishability of orthogonal product basis [2, 11].

In analogy to the quantum deficit zero-way $\Delta^\emptyset$, we can define the classical deficit zero-way $\Delta^\emptyset_{cl}$.

Definition 4. *The classical deficit zero-way $\Delta^\emptyset_{cl}(\varrho_{AB})$ of a state ϱ_{AB} is the difference between the information $I_l^\emptyset$ localizable by the zero-way protocol $\mathcal{P}^\emptyset$ and the local information*

$$\Delta^\emptyset_{cl}(\varrho_{AB}) = I_l^\emptyset(\varrho_{AB}) - I_{LO}(\varrho_{AB}), \tag{15}$$

or equivalently we have that

$$\Delta^\emptyset_{cl} = I_M - \Delta^\emptyset. \tag{16}$$

The classical deficit zero-way tells us about the amount of information that can be localized into a subsystem from correlations which are not destroyed after local measurement (or dephasing). Note that after measurement a state changes into classically correlated one (1) and then all information present in correlations of such a state can be localized.

There is a question of how great can be the quantum deficit. We know that for a pure state ψ it is given by

$$\Delta(\psi) = \tfrac{1}{2} I_M(\psi). \tag{17}$$

So in particular for the maximally entangled state ψ^+ we have $\Delta(\psi^+) = \tfrac{1}{2} I_M(\psi^+)$. Additionally for pure states $\Delta^\emptyset = \Delta$, because the greatest value of localizable information is equal to the information which can be concentrated on a subsystem using $\mathcal{P}^\emptyset$ protocol. We can ask if there exist mixed states for which the rate of quantum deficit to I_M is greater than for maximally entangled states, so if there are states for which the amount of quantumness of correlations is greater than $\tfrac{1}{2} I_M$. We are able to answer this question for quantum deficit zero-way. What is more we can find such states for which $\Delta^\emptyset = I_M$. It implies immediately that for these states $\Delta^\emptyset_{cl} = 0$. This means that after optimal local measurement (or dephasing) all correlations are destroyed, so by $\mathcal{P}^\emptyset$ protocol we cannot extract any additional information than in the local scenario.

In figure 1, we can see that there exist states for which $\Delta^\emptyset$ is equal or almost equal to I_M. So there is a task to find such states for which this equality holds.

4. States for which $\Delta^\emptyset = I_M$

In this section, we introduce a class of states ϱ_a parameterized with a and prove that these states up to some value of parameter a fulfil the condition that $\Delta^\emptyset(\varrho_a) = I_M(\varrho_a)$.

4.1. Defining the class of states

Let us define the class of states ϱ_a for $a \in [0, 1]$ on the $2 \otimes 2$ Hilbert space as

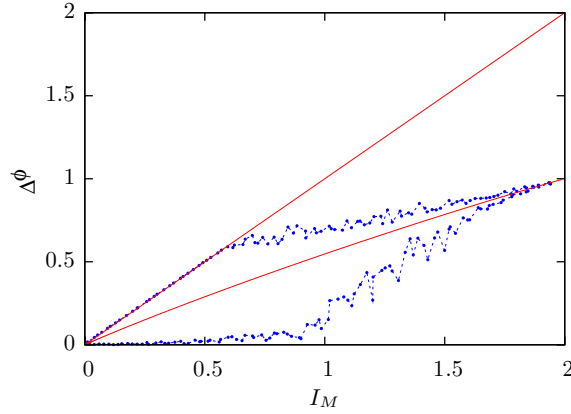$$\varrho_a = \tfrac{1}{2} \left(|\phi_a\rangle\langle\phi_a| + |\psi_a\rangle\langle\psi_a| \right), \tag{18}$$

Figure 1. Deficit zero-way is plotted versus mutual information for 100 000 random two-qubit states, only the maximal and minimal values of $\Delta^{\emptyset}$ are plotted in short intervals of I_M. The upper line is an upper bound for $\Delta^{\emptyset}$, while the lower one is $\Delta^{\emptyset}$ of isotropic states.

where

$$|\phi_a\rangle = \sqrt{a}|00\rangle + \sqrt{1-a}|11\rangle, \tag{19}$$

$$|\psi_a\rangle = \sqrt{a}|01\rangle + \sqrt{1-a}|10\rangle. \tag{20}$$

In our problem ϱ_a and ϱ_{1-a} are equivalent (they only differ by a local operation), thus we will only consider $a \in \left[0, \frac{1}{2}\right]$.

The mutual information for states ϱ_a is given by

$$I_M(\varrho_a) = H_2(a), \tag{21}$$

thus the value of I_M may be freely chosen in the interval $[0, 1]$ by a proper selection of parameter a, where H_2 denotes the binary entropy.

The deficit zero-way for states ϱ_a is given by

$$\Delta^{\emptyset}(\varrho_a) = \inf_{\Lambda \in \mathcal{P}^{\emptyset}} S(\varrho_a') - 1. \tag{22}$$

Numerical deficit zero-way versus mutual information for the states of our class ϱ_a suggests (figure 2) that for some interval $[0, a_0]$ of the parameter a we have equality between quantities $\Delta^{\emptyset}$ and I_M.

4.2. Sketch of the proof

To optimize the value of $\Delta^{\emptyset}$ given by (22) we start from the formula for $S(\varrho')$ reachable under $\mathcal{P}^{\emptyset}$ protocols. Then we reduce the optimization of $S(\varrho')$ to the optimization of single real parameter function $S_{\alpha}(\gamma_A)$, where α is determined by a. Later by analyzing the first- and second-order derivatives of $S_{\alpha}(\gamma_A)$ we argue that for $\alpha \in \left[\alpha_0, \frac{\pi}{2}\right]$ corresponding to $a \in [0, a_0]$ there is the global minimum of $S_{\alpha}(\gamma_A)$ in $\gamma_A = 0$ which implies $\Delta^{\emptyset} = I_M$.

On the other hand, for $\alpha \in [0, \alpha_0)$ corresponding to $a \in \left(a_0, \frac{1}{2}\right]$ there is a maximum of $S_{\alpha}(\gamma_A)$ in $\gamma_A = 0$ which implies $\Delta^{\emptyset} < I_M$. Finally, we compute the value of a_0.

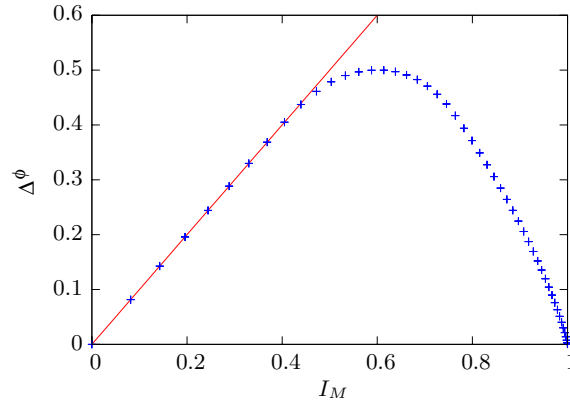**Figure 2.** Deficit zero-way is plotted versus mutual information for the class of states ϱ_a. The line is an upper bound for $\Delta^{\emptyset}$.

4.3. Helpful functions

First we introduce two functions, which will be useful in the further consideration. Let us define

$$H_s(x) = H_2\left(\frac{1 + \sin(x)}{2}\right), \tag{23}$$

$$H_c(x) = H_2\left(\frac{1 + \cos(x)}{2}\right). \tag{24}$$

The above functions have the following properties:

  (i) H_s and H_c are even so their first derivatives are odd and the second are even.
 (ii) H_s and H_c are periodic with period π and so are the derivatives.
(iii) $H_s\left(\frac{\pi}{2} \pm x\right) = H_c(x)$ and similar for the derivatives.

4.4. Simplifying the form of $S(\varrho_a')$

For any two-qubit state ϱ_a' reachable under $\mathcal{P}^{\emptyset}$ protocol, its von Neumann entropy can be expressed by

$$S(\varrho_a') = H\big(\mathrm{diagonal}\big(U_A \otimes U_B \varrho_a U_A^{\dagger} \otimes U_B^{\dagger}\big)\big), \tag{25}$$

where H denotes the Shannon entropy and U_A and U_B are single-qubit unitaries which can be parameterized as follows [12]:

$$U = e^{i\alpha}\begin{bmatrix} e^{i(-\frac{\beta}{2} - \frac{\delta}{2})}\cos\left(\frac{\gamma}{2}\right) & -e^{i(-\frac{\beta}{2} + \frac{\delta}{2})}\sin\left(\frac{\gamma}{2}\right) \\ e^{i(\frac{\beta}{2} - \frac{\delta}{2})}\sin\left(\frac{\gamma}{2}\right) & e^{i(\frac{\beta}{2} + \frac{\delta}{2})}\cos\left(\frac{\gamma}{2}\right) \end{bmatrix}. \tag{26}$$

Using this parameterization for U_A and U_B, we can simplify with the help of Maxima[3] the form of $S(\varrho_a')$ to

$$S(\varrho_a') = H\left(\frac{1 + s - c}{4}, \frac{1 - s - c}{4}, \frac{1 - s + c}{4}, \frac{1 + s + c}{4}\right), \tag{27}$$

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[3] Maxima (http://maxima.sourceforge.net/) is a free software computer algebra system released under GNU GPL.

where

$$s = 2\sqrt{a(1-a)} \sin\gamma_A \sin\gamma_B \cos\delta_A \cos\delta_B, \tag{28}$$

$$c = (1-2a)\cos\gamma_A. \tag{29}$$

(We subscript the parameters with the name of the subsystem.) Note that this form of $S(\varrho'_a)$ exhibits asymmetry with respect to the parameters of U_A and U_B. Our class of states ϱ_a is indeed asymmetric (i.e. $\varrho_a \neq V\varrho_a V$ where V is the swap operator) unless $a = \frac{1}{2}$, for example, $S_A(\varrho_a) \neq S_B(\varrho_a)$. The asymmetry with respect to the parameters of U_A and U_B will appear even stronger after the next simplification.

Let us here recall the following fact.

Fact 1. *Any change toward equalization of probabilities $p_1, p_2, \ldots, p_n$ increases H. Thus if $p_1 < p_2$ and we increase p_1, decreasing p_2 by an equal amount so that p_1 and p_2 are more nearly equal, then H increases (...) [13], i.e. the Shannon entropy is a Schur-concave function.*

We group probabilities from distribution (27) in two inequalities

$$\frac{(1 \pm c) - |s|}{4} \leqslant \frac{(1 \pm c) + |s|}{4}. \tag{30}$$

Now we can see that decreasing $|s|$ for any fixed value of c will change both inequalities towards equalization and hence from fact 1 will increase entropy (27). As we are minimizing $S(\varrho'_a)$ in (22), we should maximize $|s|$ with respect to the parameters with no influence on c. We can do this by setting

$$\sin\gamma_B \cos\delta_A \cos\delta_B = 1 \tag{31}$$

in (28). After this substitution we observe that $s + c$ and $s - c$ are harmonic oscillations, i.e.

$$s \pm c = \sin(\gamma_A \pm \alpha), \tag{32}$$

where α is determined by

$$\sin\alpha = 1 - 2a, \tag{33}$$

$$\cos\alpha = 2\sqrt{a(1-a)}. \tag{34}$$

The harmonic oscillations of (32) allow us to simplify $S(\varrho'_a)$ given by (27) to

$$S_\alpha(\gamma_A) = 1 + \tfrac{1}{2}H_s(\gamma_A + \alpha) + \tfrac{1}{2}H_s(\gamma_A - \alpha) \tag{35}$$

with the property that

$$\inf_{\Lambda \in \mathcal{P}^\emptyset} S(\varrho'_a) = \inf_{\gamma_A \in [-\frac{\pi}{2}, \frac{\pi}{2}]} S_\alpha(\gamma_A). \tag{36}$$

4.5. Proving $\Delta^\emptyset = I_M$ for $a \in [0, a_0]$

First we observe that

$$S_\alpha(0) = 1 + H_2(a), \tag{37}$$

thus if $\gamma_A = 0$ is the global minimum of $S_\alpha(\gamma_A)$ for some α we get the desired equality

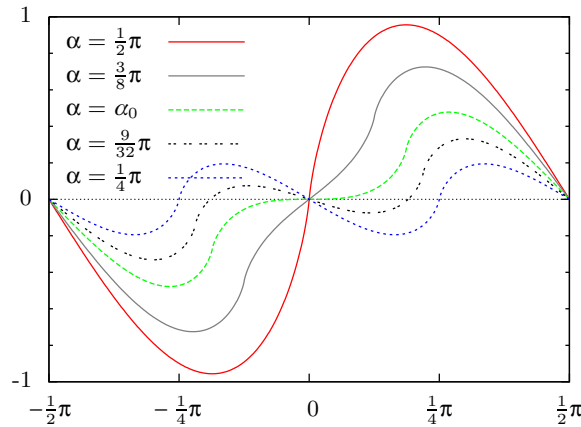$$\Delta^\emptyset(\varrho_a) = (1 + H_2(a)) - 1 = I_M(\varrho_a). \tag{38}$$

Figure 3. First-order derivative of $S_\alpha(\gamma_A)$ for sample values of α illustrating the appearance of two more zeros.

On the other hand, if $\gamma_A = 0$ is a maximum of $S_\alpha(\gamma_A)$ for some α then the value of $S_\alpha(\gamma_A)$ in the global minimum is less than $S_\alpha(0)$ and thus $\Delta^\emptyset < I_M$.

In search for the extrema of $S_\alpha(\gamma_A)$, we consider its derivative

$$\frac{dS_\alpha(\gamma_A)}{d\gamma_A} = \frac{1}{2}\left(\frac{dH_s(\gamma_A + \alpha)}{d\gamma_A} + \frac{dH_s(\gamma_A - \alpha)}{d\gamma_A}\right). \tag{39}$$

Note that $S_\alpha(\gamma_A)$, same as H_s, is periodic with period π. Thus, it is enough to consider the range of a single period $\gamma_A \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We identify both ends of the period when it comes to listing zeros in a period. In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the derivative of $S_\alpha(\gamma_A)$ has for all α zeros in $\gamma_A = 0$ and $\gamma_A = \frac{\pi}{2}$ (as the derivatives of H_s and H_c are odd; figure 3).

We also consider the second-order derivative of $S_\alpha(\gamma_A)$ which is the average of two *impulses* relatively shifted by 2α. By the *impulse* we mean a function of the form

$$\frac{d^2 H_s(x)}{dx^2} = \frac{1}{2}\sin(x) \cdot (\log \circ f \circ \sin)(x) - \frac{1}{\ln 2}, \tag{40}$$

where

$$f(x) = \frac{1+x}{1-x}. \tag{41}$$

Since f is strictly increasing in the interval $[-1, 1)$, thus $f \circ \sin$ is strictly increasing in $\left[0, \frac{\pi}{2}\right)$ and strictly decreasing in $\left(-\frac{\pi}{2}, 0\right]$, and the same holds for $\log \circ f \circ \sin$ and the impulse. The impulse has a *peak* at $\frac{\pi}{2}$ by which we mean it tends to infinity at this point. The impulse also has one negative and one positive interval per period. It can be observed in figure 4 as the curve labeled $\alpha = 0$.

Now we analyze the extrema of $S_\alpha(\gamma_A)$ for $\alpha \in \left[0, \frac{\pi}{2}\right]$, which corresponds to $a \in \left[0, \frac{1}{2}\right]$.

For $\alpha = \frac{\pi}{2}$ corresponding to $a = 0$ both impulses of the second-order derivative of $S_\alpha(\gamma_A)$ are equal and have peaks at $\gamma_A = 0$ (figure 4) and thus $\gamma_A = 0$ is a minimum of $S_\alpha(\gamma_A)$. If we move with α from $\frac{\pi}{2}$ down to 0 then the peaks of both impulses are shifting away from $\gamma_A = 0$, and thus the value of the second-order derivative of $S_\alpha(\gamma_A)$ in $\gamma_A = 0$ is strictly monotonically decreasing to $(-\ln 2)^{-1}$ for $\alpha = 0$. The point at which it reaches 0 on this way will be called α_0.
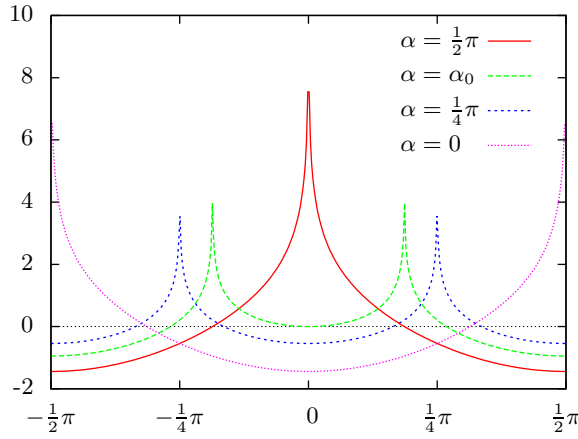
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Figure 4. Second-order derivative of $S_\alpha(\gamma_A)$ for the most characteristic values of the range $\alpha \in [0, \frac{\pi}{2}]$.

Thus for $\alpha \in [0, \alpha_0)$ corresponding to $a \in \left(a_0, \frac{1}{2}\right]$ the second-order derivative is negative in $\gamma_A = 0$, and so $S_\alpha(\gamma_A)$ has the maximum in $\gamma_A = 0$ which implies $\Delta^\emptyset < I_M$.

For $\alpha \in \left[\alpha_0, \frac{\pi}{2}\right]$ corresponding to $a \in [0, a_0]$ the second-order derivative of $S_\alpha(\gamma_A)$ has one negative and one positive interval per period, which implies that the first-order derivative has only one strictly increasing and one strictly decreasing interval per period and thus at most two zeros (one per interval). So it has exactly two zeros, $\gamma_A = 0$ and $\gamma_A = \frac{\pi}{2}$, which must appear for all α. For $\alpha \in \left[\alpha_0, \frac{\pi}{2}\right]$ the first-order derivative of $S_\alpha(\gamma_A)$ is negative in $\gamma_A = \frac{\pi}{2}$, thus $\gamma_A = \frac{\pi}{2}$ is the global maximum of $S_\alpha(\gamma_A)$ which implies $\gamma_A = 0$ is the global minimum and so $\Delta^\emptyset = I_M$ for $a \in [0, a_0]$.

The last step is to find a_0.

4.6. Finding a_0

To find a_0 we numerically solve the equation

$$\left.\frac{\mathrm{d}^2 S_\alpha(\gamma_A)}{\mathrm{d}\gamma_A^2}\right|_{\gamma_A=0} = (1 - 2a)\mathrm{arctanh}(1 - 2a) - 1 = 0, \tag{42}$$

and the smaller of the two roots gives a_0:

$$a_0 \approx 0.083\,22. \tag{43}$$

5. Reaching lower bound on $\Delta^\emptyset$ for a given I_M

On the $2 \otimes 2$ Hilbert space $\Delta_{cl}^\emptyset \leqslant 1$, thus on this space the lower bound on $\Delta^\emptyset$ is a function of I_M

$$\Delta^\emptyset \geqslant \max(0, I_M - 1). \tag{44}$$

We show that for any I_M this lower bound is achievable.

For $I_M \in [0, 1]$ the lower bound is achieved by the simple class of separable states

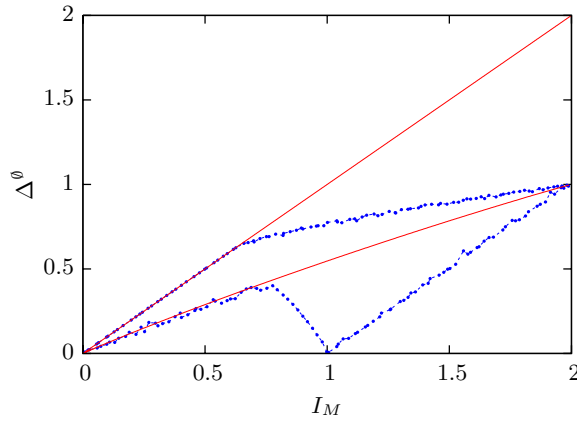$$\sigma_p = p|00\rangle\langle00| + (1 - p)|11\rangle\langle11|. \tag{45}$$

Figure 5. Deficit zero-way is plotted versus mutual information for 10 000 random two-qubit states from $\varrho_{a,b,p}$ class, only the maximal and minimal values of $\Delta^{\emptyset}$ are plotted in short intervals of I_M. The upper line is an upper bound for $\Delta^{\emptyset}$, while the lower one is $\Delta^{\emptyset}$ of isotropic states.

While for $I_M \in [1, 2]$ the lower bound is achieved by the class of states

$$\varrho_p = p|\phi^+\rangle\langle\phi^+| + (1 - p)|\psi^+\rangle\langle\psi^+|, \tag{46}$$

where

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{47}$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \tag{48}$$

For σ_p the infimum of $\Delta^{\emptyset}$ as given by (13) is achieved by setting $U_A = U_B = I$ in (25) while for ϱ_p by $U_A = U_B = H$, where H is the Hadamard gate.

6. Another interesting class

Let us introduce another class of states on the $2 \otimes 2$ Hilbert space

$$\varrho_{a,b,p} = p|\phi_a\rangle\langle\phi_a| + (1 - p)|\psi_b\rangle\langle\psi_b|, \tag{49}$$

where $|\phi_a\rangle$ and $|\psi_b\rangle$ are given by (19) and (20), respectively. This class is a generalization of two of the previously considered classes, i.e.

$$\varrho_a = \varrho_{a,a,\frac{1}{2}}, \tag{50}$$

$$\varrho_p = \varrho_{\frac{1}{2},\frac{1}{2},p}. \tag{51}$$

Random elements of the $\varrho_{a,b,p}$ class (figure 5) cover wider range of $\Delta^{\emptyset}$ than the general two-qubit states (figure 1), except some evident region, although there are 100 000 states in figure 1 and only 10 000 in figure 5.

What is interesting for $\varrho_{a,b,p}$ states the condition $\Delta^{\emptyset} = I_M$ require $\varrho_{a,b,p}$ to be a separable state. It comes from the fact that if we want to fulfil this condition then after the measurement is made in the eigenbasis of subsystems of a state we have to obtain a product state. Otherwise, we would be able to get more than $2 - S_A - S_B$ localizable information zero-way, because

$S(\varrho') < S_A + S_B$ for nonproduct states. The above condition for states $\varrho_{a,b,p}$ can be expressed by the parameters a, b, p as

$$p^2 a(a-1) - (1-p)^2 b(1-b) = 0, \tag{52}$$

which is equivalent to

$$p\sqrt{a(a-1)} = (1-p)\sqrt{b(1-b)}. \tag{53}$$

And equality (53) implies via partial transposition criterion that $\varrho_{a,b,p}$ is a separable state.

7. Entangled states for which $\Delta^\emptyset = I_M$ may hold

Consider the following class of states (mixtures of two nonorthogonal states) on the $2 \otimes 2$ Hilbert space

$$\sigma_a = \frac{1}{2} \left(\frac{|u\rangle\langle u|}{\|u\|} + \frac{|v_a\rangle\langle v_a|}{\|v_a\|} \right), \tag{54}$$

where

$$|u\rangle = |00\rangle + 2|11\rangle, \tag{55}$$

$$|v_a\rangle = |00\rangle + a|01\rangle - 2|10\rangle - 2|11\rangle. \tag{56}$$

For the state σ_a to have product diagonal in the eigenbasis of its subsystems (which is the necessary condition of $I_M = \Delta^\emptyset$ as shown in the previous section), the parameter a must satisfy the equation

$$(4a^4 + 40a^3 + 87a^2 + 160a - 341)R(a) = 0, \tag{57}$$

where $R(a)$ is some rational expression. The polynomial of degree 4 has two real solutions, one around -8.1 and the other around 1.1. Both give entangled states but the negative one has $I_M - \Delta^\emptyset > 0.09$. The positive one is

$$a_0 = \frac{\sqrt{7200\sqrt{z} - \sqrt{w}\left(w - 504z^{\frac{1}{3}}\right)}}{8w^{\frac{1}{4}}z^{\frac{1}{6}}} - \frac{\sqrt{w}}{8z^{\frac{1}{6}}} - \frac{5}{2} \tag{58}$$

$$\approx 1.101\,22, \tag{59}$$

where

$$w = 16z^{\frac{2}{3}} + 168z^{\frac{1}{3}} - 3111, \tag{60}$$

$$z = \frac{75\sqrt{527\,523}}{16} - \frac{131\,787}{64}. \tag{61}$$

The state σ_{a_0} is entangled and satisfies the required condition of equality, and numerical optimizations and parameterized plots suggest it may fulfil the $I_M = \Delta^\emptyset$ equality.

If we generalize σ_a to $\sigma_{a,t}$, where instead of $|v_a\rangle$ we take

$$|v_{a,t}\rangle = |00\rangle + a|01\rangle - t|10\rangle - t|11\rangle, \tag{62}$$

it seems that for all t starting with some t_0 ($1 < t_0 < 2$) we can find a_t with product diagonal in the eigenbasis of subsystems and $I_M = \Delta^\emptyset$ (or almost equal). So $\sigma_{a_t,t}$ may be (for some range of t) a class of entangled states satisfying the equality of $I_M = \Delta^\emptyset$.

8. Summary

In our paper, we showed that there are states for which quantum correlations are completely quantum. We made this by presenting a family of states for which mutual information (the measure of total correlations) is equal to quantum deficit zero-way (the measure of quantumness of correlations). Surprisingly, the states which we have found are separable. We also presented numerical results which suggest that such a situation is possible for entangled states. What is intriguing we know that the optimal protocol $\mathcal{P}^{\emptyset}$ which achieves the value of mutual information is made in the local eigenbasis of subsystems and gives as a result a product state of the form $\varrho_A \otimes \varrho_B$, where ϱ_A and ϱ_B are states of subsystems. This is equivalent to localizing only the local information. Any other local measurement gives us nonproduct classical states, from which we are able to localize whole of its global information but still less than by only local action. So the price of producing classical correlations from which we can get additional information is too high. The produced entropy is greater than the gained information.

Acknowledgments

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